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SLIDING REGIMES AND ANISOTROPY IN OPTIMAL DESIGN OF VIBRATING AXISYMMETRIC PLATES†

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Abstract—This paper deals with optimal design of solid, elastic, axisymmetric plates performing free, transverse vibrations. It is the objective to determine the plate thickness distribution from the condition that the plate volume is minimized for a given value of the fundamental natural frequency, or for a given higher order natural frequency that corresponds to a vibration mode with a prescribed number of nodal diameters.

It is found that the Weierstrass necessary condition for optimality is generally not satisfied for a traditional formulation of this problem, and that the optimal design is characterized by a sliding regime of control where the plate thickness exhibits an infinite number of discontinuities, as a system of infinitely thin, circumferential stiffeners are formed on the optimal axisymmetric plate. This inherent anisotropy of the optimal design is taken into account in a regularization of the initial optimization problem by establishing the tensorial character of the plate bending rigidity and using the concentration of thin, circumferential stiffeners as a new design variable (control). It is shown that the new formulation of the problem can be solved numerically, and examples of optimal designs are presented in the paper.

INTRODUCTION

We consider the problem of determining the optimal thickness distribution of a solid, axisymmetric Kirchhoff plate, which for given plate radius, boundary conditions and material, provides a minimum plate volume under a prescribed value of the fundamental frequency for free, transverse vibrations. This problem and the equivalent dual problem of maximizing the frequency for specified volume of the plate, have already been considered in several papers[1–4, 6, 7].

The axisymmetric problem was first treated in geometrically unconstrained form, i.e. without specification of minimum and maximum allowable values for the thickness distribution[1–4]. Reference[1] presents, for three classical sets of boundary conditions, numerical solutions obtained on the basis of the so-called optimality condition, which, strictly speaking, is only a stationarity condition. The solution for a simply supported plate in[1] was later reproduced in[2], and it was subsequently shown mathematically[3] that the solution satisfies the Legendre necessary condition for optimality, but that it represents a local optimum.

In fact, it is shown in[4] that whereas a number of local optimal solutions exist to the geometrically unconstrained problem, it does not possess a global optimal solution: in the absence of some positive lower bound on the thickness, the latter can be made vanish at arbitrarily chosen sets of concentric circles (where hinges develop) in the plate, and each corresponding thickness distribution would constitute a local optimal solution. By increasing the number of such circles, high and very thin concentric stiffeners arise between them, and the plate volume can be rendered arbitrarily small for fixed fundamental frequency. Similarly, it is earlier shown in [5] that concentration of part of a given volume within high and thin stiffeners can yield any desired increase of the fundamental frequency in the absence of an upper bound on the thickness.

It is suggested in[6] that a global optimal solution might be ensured by specifying both a minimum and a maximum allowable value for the plate thickness in connection with the traditional optimal design formulation. However, even this remedy is not sufficient[7, 8].

†Dedicated to the memory of Prof. Anatolii I. Lurie.

For example, considering such a geometrically constrained formulation, Armand[7] found a conventional iterative computational procedure to be divergent, and that rapidly varying stiffener-like thickness distributions occurred. Armand reports that "the zigzag profile exhibited presents a number of peaks and valleys, the number and location of which vary greatly with the slightest changes in the initial conditions". His conclusion is that "such a behaviour seems to indicate the existence of more than one solution to the optimization problem in the case of a plate; the shape of different nominal solutions obtained in solving the governing equations also suggests that part of the weight tends to concentrate along discrete stiffeners, although convergence cannot be acheived".

However, in a recent paper[8] on the similar problem of geometrically constrained optimal design of solid, elastic plates for minimum compliance (maximum stiffness) under fixed plate volume, the investigators were able to obtain convergent numerical results by using an incremental procedure of "optimal structural remodeling"[9]. But in this case, solutions associated with moderate to large ratios between the maximum and minimum allowable values for the plate thickness are found to depend significantly on the number of grid points used in the numerical solution procedure; solving a given problem by using a sequence of finer and finer grids, increasing numbers of thinner and thinner integral stiffeners are formed. At the same time, the compliance is decreasing for the sequence of fixed volume designs obtained. It is found in[8] that although a number of local optimal solutions can be obtained to the traditional optimal design formulation, it is not possible to determine a possible global optimal solution within this formulation by using a finite number of grid points. The conclusion drawn in[8] is that the global optimal design, if it exists, must be a plate which, at least in some sub-regions, is equipped with an infinite number of infinitely thin stiffeners. This has proven to be correct in a very recent follow-up[10] of Ref.[8].

The phenomena described above implicitly witness that optimal plate designs are characterized by sliding regimes of control. On the other hand, it has earlier been shown in [11] that the traditional formulation of the general two-dimensional problem of maximizing the fundamental frequency of a plate does not possess an optimal solution since the Weierstrass necessary condition breaks down at almost each point of the plate if strong, local thickness variations are admitted. At the same time, it directly follows from[11] that at those points where the curvature tensor considerably differs from a circular one, the Legendre condition seizes to be satisfied, which demonstrates non-optimality of a stationary solution even if thickness constraints are considered.

The failure of Weierstrass and Legendre tests in traditionally formulated, general 2-D plate optimization problems explains the appearance of different kinds of degeneracies, and at the same time manifests the possibility of sliding optimal regimes of control. It will be shown in the following section that for the simpler, essentially 1-D problem of optimizing an axisymmetric plate, the necessary condition of Weierstrass is also not satisfied in general.

1. NECESSARY CONDITIONS OF OPTIMALITY, TRADITIONAL FORMULATION

Consider small, free harmonic vibrations at the fundamental angular frequency ω of a thin, solid, elastic, axisymmetric circular plate with simply supported boundary. If we denote the distance from the plate center by r and assume the fundamental vibration mode w to be axisymmetric, then $w = w(r)$ will satisfy the following system of differential equations

$$\begin{aligned} w' &= p \\ p' &= \frac{M}{h^3} - \frac{\nu}{r} p \\ M' &= -S - \frac{1-\nu}{r} M + \frac{1-\nu^2}{r^2} h^3 p \\ S' &= -\frac{1}{r} S - \omega^2 h w \end{aligned} \tag{1.1}$$

together with the boundary conditions

$$p(0) = S(0) = w(R) = M(R) = 0. \tag{1.2}$$

Here p denotes the slope of the deflection w, M is the radial bending moment, S the radial shear force, and h is the plate thickness function, which is considered as the design variable (control). The plate radius is denoted by R, and ν is the Poisson ratio of the material. The plate bending rigidity D is assumed to be scalar (i.e. isotropic) and is divided by the mass density of the material and set equal to h^3, which corresponds to some specific norm of the deflection function.

The fundamental angular frequency ω is assumed to be prescribed, and the volume of the plate represented by

$$V = \int_0^R hr\,\mathrm{d}r \tag{1.3}$$

is to be minimized under differential constraints (1.1) due to a suitable determination of the design variable h belonging to a given closed range

$$0 < h_{\min} \le h \le h_{\max} \tag{1.4}$$

for geometrically constrained optimal design, or open range

$$0 \le h \tag{1.5}$$

for geometrically unconstrained design, respectively.

The formulation outlined above, where h is used as the design variable and the bending rigidity is assumed to be isotropic, is termed "traditional formulation" for optimal design in the present paper. This problem is self-adjoint, and the necessary condition of stationarity of V for arbitrary admissible variation of h is [1, 2]

$$\frac{\partial H}{\partial h} = \left\{3h^2\left[\left(\frac{M}{h^3}\right)^2 + \frac{1-\nu^2}{r^2}p^2\right] - \omega^2 w^2 - \frac{1}{c^2}\right\}r = 0 \tag{1.6}$$

where c^2 is a positive constant which depends on the norm of the deflection w. Condition (1.6) is valid in all subintervals of $0 \le r \le R$ where h satisfies the inequalities $h_{\min} < h < h_{\max}$ or $0 < h$, see (1.4) and (1.5), respectively.

In eqn (1.6), H represents the Hamiltonian of the problem, and the Legendre necessary condition for minimum volume V is $\partial^2 H/\partial h^2 \le 0$, i.e.

$$-2\frac{M^2}{h^5} + \frac{1-\nu^2}{r^2}p^2 h \le 0. \tag{1.7}$$

It follows directly from this condition that a stationary distribution of the plate thickness will be non-optimal, if, at a point of vanishing radial bending moment M, the thickness is intermediate in a geometrically constrained problem, or nonvanishing in a geometrically unconstrained one.

In Ref. [3], the Legendre necessary condition is shown to be satisfied throughout for a smooth solution first obtained in [1] and later in [2] to a geometrically unconstrained formulation of the simply supported plate optimization problem†, but this solution is shown to represent a local minimum for the plate volume [3].

†Corresponding solutions published in [1] for a clamped plate and for a centrally supported plate with a free edge also satisfy the Legendre condition (1.7) throughout. This is easily checked by means of numerical listings of h, M and p as functions of r available in [12] for these solutions (Ref. [12] provides a basis for [1]).

Now, if we admit strong, local variations of h, the increment

$$\Delta V = V(h+\Delta h) - V(h) = \int_0^R \Delta f r \, dr \tag{1.8}$$

of V in eqn (1.3) can be calculated as in[11], and is given via (1.8) by

$$\begin{aligned}\Delta f = &-\Delta h\left\{-1 - c^2\omega^2 w^2 + 3c^2h^2\left[\left(\frac{M}{h^3}\right)^2 + \frac{1-\nu^2}{r^2}p^2\right]\right\}\\ &+(\Delta h)^2\frac{c^2}{(h+\Delta h)^3}\left\{M^2\left[6+8\frac{\Delta h}{h}+3\left(\frac{\Delta h}{h}\right)^2\right]h^{-2}\right.\\ &\left.-\frac{1-\nu^2}{r^2}p^2\left[3+10\frac{\Delta h}{h}+12\left(\frac{\Delta h}{h}\right)^2+6\left(\frac{\Delta h}{h}\right)^3+\left(\frac{\Delta h}{h}\right)^4\right]h^2\right\},\end{aligned} \tag{1.9}$$

where Δh denotes the difference between an admissible and optimal distribution of the design variable. In eqn (1.8), the coefficient of Δh vanishes due to the stationarity condition (1.6), and it can then directly be seen from the remaining nonlinear term that the Weierstrass necessary condition for optimality $\Delta f \geq 0$ fails to be satisfied everywhere provided that Δh is large enough.

2. THE EQUIVALENT ANISOTROPIC PLATE BENDING RIGIDITY TENSOR

The phenomena described above and recent results obtained in [8] make it necessary to reformulate the present problem since the optimal thickness function may well constitute an infinitely fast sequence of infinitely thin layers of different heights perpendicular to the plate mid plane. Assuming sequences of such kind to be admissible, we shall now establish the bending rigidity tensor $D_{\alpha\beta\kappa\gamma}$ which characterizes our anisotropic plate.

To determine this tensor, consider a small rectangular element σ within the plate domain. This element, which is shown in Fig. 1, represents a physically small material volume; it is small in comparison with the plate domain, and the stress–strain field within it is homogeneous up to the order $o(\text{mes}\,\sigma)$, but the element is large in comparison with its microstructure. The element consists of a finite number of parallel layers of material perpendicular to the plate mid-plane with rigidity tensors $D^+_{\alpha\beta\kappa\gamma}$ and $D^-_{\alpha\beta\kappa\gamma}$ associated with the thicknesses h_+ and h_- ($h_- < h_+$) over the domains σ_+ and σ_-, respectively. We introduce the concentration μ

$$0 \leq \mu = \frac{\text{mes}\,(\sigma_+)}{\text{mes}\,(\sigma)} \leq 1 \tag{2.1}$$

of the domain σ_+ of thickness h_+ within the element σ.

Now define homogeneous curvatures $e^+_{\alpha\beta}$ and $e^-_{\alpha\beta}$ and bending moments $M^+_{\alpha\beta}$ and $M^-_{\alpha\beta}$ within the domains σ_+ and σ_-, respectively,

$$\begin{aligned}M^+_{\alpha\beta} &= D^+_{\alpha\beta\kappa\gamma}e^+_{\kappa\gamma}\\ M^-_{\alpha\beta} &= D^-_{\alpha\beta\kappa\gamma}e^-_{\kappa\gamma}.\end{aligned} \tag{2.2}$$

Here, the Greek indices refer to the axes of a Cartesian coordinate system x_1x_2 embedded in the plate mid plane, see Fig. 1, and repeated indices imply summation. The average moment tensor $M_{\alpha\beta}$ and average deformation tensor $e_{\alpha\beta}$ within the element σ are given by

$$\begin{aligned}M_{\alpha\beta} &= \mu M^+_{\alpha\beta} + (1-\mu)M^-_{\alpha\beta}\\ e_{\alpha\beta} &= \mu e^+_{\alpha\beta} + (1-\mu)e^-_{\alpha\beta}.\end{aligned} \tag{2.3}$$

The continuity conditions at the boundaries between the σ_+ and σ_- domains express the continuity of the normal component of the bending moment and the tangential components of

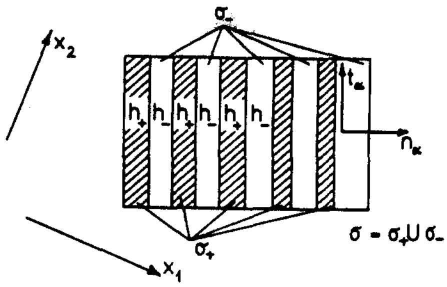

Fig. 1. Small, rectangular element σ of the plate domain. The element consists of thin, parallel layers of different heights h_+ and h_- perpendicular to the plate mid plane. The unions of subdomains with layers of heights h_+ and h_- are denoted by σ_+ and σ_-, respectively.

the curvatures, i.e.

$$\begin{aligned} M^+_{\alpha\beta} n_\alpha n_\beta &= M^-_{\alpha\beta} n_\alpha n_\beta \\ e^+_{\alpha\beta} n_\alpha t_\beta &= e^-_{\alpha\beta} n_\alpha t_\beta \\ e^+_{\alpha\beta} t_\alpha t_\beta &= e^-_{\alpha\beta} t_\alpha t_\beta, \end{aligned} \tag{2.4}$$

where the directions of the unit vectors n_α and t_α are perpendicular and parallel to the boundaries, respectively, see Fig. 1. Equations $(2.4)_2$, $(2.4)_3$ and $(2.3)_2$ show that the curvatures in the σ_+ and σ_- domains only differ by their normal components, which may be expressed as

$$\begin{aligned} e^+_{\alpha\beta} &= e_{\alpha\beta} + a_+ n_\alpha n_\beta \\ e^-_{\alpha\beta} &= e_{\alpha\beta} + a_- n_\alpha n_\beta, \end{aligned} \tag{2.5}$$

where a_+ and a_- are scalars to be determined subsequently. Firstly, substitute eqns (2.5) into eqn $(2.3)_2$. Secondly, substitute eqns (2.2) into eqn $(2.4)_1$, and eliminate then $e^+_{\alpha\beta}$ and $e^-_{\alpha\beta}$ by means of eqns (2.5). Hence, we obtain

$$\begin{aligned} &\mu a_+ + (1-\mu) a_- = 0 \\ &D^+_{\alpha\beta\kappa\gamma}(e_{\kappa\gamma} + a_+ n_\kappa n_\gamma) n_\alpha n_\beta - D^-_{\alpha\beta\kappa\gamma}(e_{\kappa\gamma} + a_- n_\kappa n_\gamma) n_\alpha n_\beta = 0. \end{aligned} \tag{2.6}$$

Solving these two linear equations with respect to a_+ and a_-, we find

$$\begin{aligned} a_+ &= \frac{1-\mu}{\tilde{D}} (D^-_{\alpha\beta\kappa\gamma} - D^+_{\alpha\beta\kappa\gamma}) n_\alpha n_\beta e_{\kappa\gamma} \\ a_- &= -\frac{\mu}{\tilde{D}} (D^-_{\alpha\beta\kappa\gamma} - D^+_{\alpha\beta\kappa\gamma}) n_\alpha n_\beta e_{\kappa\gamma}, \end{aligned} \tag{2.7}$$

where the scalar $\tilde{D}$ is defined by

$$\tilde{D} = \{(1-\mu) D^+_{\alpha\beta\kappa\gamma} + \mu D^-_{\alpha\beta\kappa\gamma}\} n_\alpha n_\beta n_\kappa n_\gamma. \tag{2.8}$$

The curvature tensors $e^+_{\alpha\beta}$ and $e^-_{\alpha\beta}$ are consequently expressed by means of the average curvature tensor $e_{\alpha\beta}$ as

$$\begin{aligned} e^+_{\alpha\beta} &= \left[\delta_{\alpha\kappa}\delta_{\beta\gamma} + \frac{1-\mu}{\tilde{D}} (D^-_{\xi\eta\kappa\gamma} - D^+_{\xi\eta\kappa\gamma}) n_\xi n_\eta n_\alpha n_\beta\right] e_{\kappa\gamma} \\ e^-_{\alpha\beta} &= \left[\delta_{\alpha\kappa}\delta_{\beta\gamma} - \frac{\mu}{\tilde{D}} (D^-_{\xi\eta\kappa\gamma} - D^+_{\xi\eta\kappa\gamma}) n_\xi n_\eta n_\alpha n_\beta\right] e_{\kappa\gamma}, \end{aligned} \tag{2.9}$$

where $\delta_{\alpha\beta}$ is Kronecker's delta and where $\tilde{D}$ is given by eqn (2.8).

Equations (2.9) now make it possible to determine the average bending rigidity tensor $D_{\alpha\beta\kappa\gamma}$ characterizing the properties of the entire element,

$$M_{\alpha\beta} = D_{\alpha\beta\kappa\gamma} e_{\kappa\gamma}. \tag{2.10}$$

By eqns $(2.3)_1$, (2.2), (2.5) and (2.7) we have

$$M_{\alpha\beta} = \mu M^+_{\alpha\beta} + (1-\mu) M^-_{\alpha\beta} = \mu D^+_{\alpha\beta\epsilon\theta} e^+_{\epsilon\theta} + (1-\mu) D^-_{\alpha\beta\epsilon\theta} e^-_{\epsilon\theta} = \left[\mu D^+_{\alpha\beta\kappa\gamma} + (1-\mu) D^-_{\alpha\beta\kappa\gamma} - \frac{\mu(1-\mu)}{\tilde{D}} (D^-_{\alpha\beta\epsilon\theta} - D^+_{\alpha\beta\epsilon\theta})(D^-_{\xi\eta\kappa\gamma} - D^+_{\epsilon\eta\kappa\gamma}) n_\xi n_\eta n_\epsilon n_\theta\right] e_{\kappa\gamma},$$

which, in view of eqn (2.10) and the arbitrary character of the $e_{\kappa\gamma}$ tensor, implies that the $D_{\alpha\beta\kappa\gamma}$ tensor is given by

$$D_{\alpha\beta\kappa\gamma} = \mu D^+_{\alpha\beta\kappa\gamma} + (1-\mu) D^-_{\alpha\beta\kappa\gamma} - \frac{\mu(1-\mu)}{\tilde{D}} (D^-_{\alpha\beta\epsilon\theta} - D^+_{\alpha\beta\epsilon\theta})(D^-_{\xi\eta\kappa\gamma} - D^+_{\xi\eta\kappa\gamma}) n_\xi n_\eta n_\epsilon n_\theta, \tag{2.11}$$

with $\tilde{D}$ defined in eqn (2.8).

By means of (2.11) and the well known relationships $M_{\alpha\beta} = D\{(1-\nu) w_{,\alpha\beta} + \nu \delta_{\alpha\beta} w_{,\gamma\gamma}\}$ between plate moment components $M_{\alpha\beta}$ and curvatures $w_{,\alpha\beta}$ (denoted by $e_{\alpha\beta}$ until now), it can easily be checked that the following formulas

$$\begin{aligned}
D_{1111} &= \frac{D_- D_+}{\mu D_- + (1-\mu) D_+} \\
D_{2222} &= \mu D_+ + (1-\mu) D_- - \nu^2 \frac{\mu(1-\mu)}{\mu D_- + (1-\mu) D_+} (D_- - D_+)^2 \\
D_{1212} &= D_{1221} = D_{2112} = D_{2121} = \frac{1-\nu}{2} [\mu D_+ + (1-\mu) D_-] \\
D_{1122} &= D_{2211} = \nu \frac{D_- D_+}{\mu D_- + (1-\mu) D_+}
\end{aligned} \tag{2.12}$$

express the non-vanishing components of the Cartesian $D_{\alpha\beta\kappa\gamma}$ tensor corresponding to an orthotropic plate composed of a sequence of thin, parallel layers disposed perpendicular to the x_1-direction. These layers consist of isotropic material of thicknesses h_+ and h_- perpendicular to the plate mid-plane, and we have $D_+ \propto h_+^3$ and $D_- \propto h_-^3$, respectively, for these layers.

Introducing the short-hand notations $D_{/\!/}$ and $D_\perp$ for the expressions

$$\begin{aligned}
D_{/\!/} &= \frac{D_- D_+}{\mu D_- + (1-\mu) D_+} = \left[\frac{\mu}{D_+} + \frac{1-\mu}{D_-}\right]^{-1} \\
D_\perp &= \mu D_+ + (1-\mu) D_-
\end{aligned} \tag{2.13}$$

such that we have

$$\begin{aligned}
D_{1111} &= D_{/\!/} \\
D_{2222} &= D_\perp - \nu^2 (D_\perp - D_{/\!/}) = (1-\nu^2) D_\perp + \nu^2 D_{/\!/} \\
D_{1212} &= D_{1221} = D_{2112} = D_{2121} = \frac{1-\nu}{2} D_\perp \\
D_{1122} &= D_{2211} = \nu D_{/\!/},
\end{aligned} \tag{2.14}$$

the Cartesian form of Hooke's law corresponding to (2.10) is

$$
\begin{aligned}
M_{11} &= D_{/\!/}\frac{\partial^2 w}{\partial x_1{}^2} + \nu D_{/\!/}\frac{\partial^2 w}{\partial x_2{}^2} \\
M_{22} &= [(1-\nu^2)D_{\perp} + \nu^2 D_{/\!/}]\frac{\partial^2 w}{\partial x_2{}^2} + \nu D_{/\!/}\frac{\partial^2 w}{\partial x_1{}^2} \\
M_{12} &= M_{21} = (1-\nu)D_{\perp}\frac{\partial^2 w}{\partial x_1 \partial x_2}.
\end{aligned}
\tag{2.15}
$$

Note that eqns (2.15) reduce to the Hooke's law for an isotropic plate with the (scalar) bending rigidity D_+ or D_- if we have $\mu = 1$ or $\mu = 0$, respectively.

3. THE REGULARIZED OPTIMIZATION PROBLEM

We shall now reformulate the initial optimization problem by means of the results of Section 2. Assuming the axisymmetric plate to be anisotropic (circularly orthotropic), its free, harmonic vibrations are governed by the equations

$$
\begin{aligned}
M_r &= D_r\left[\frac{\partial^2 w}{\partial r^2} + \nu_\theta\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] \\
M_\theta &= D_\theta\left[\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \nu_r\frac{\partial^2 w}{\partial r^2}\right] \\
M_{r\theta} &= D_{r\theta}\frac{\partial^2}{\partial r \partial \theta}\left(\frac{1}{r}w\right) \\
Q_r &= -\frac{\partial M_r}{\partial r} - \frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta} - \frac{M_r - M_\theta}{r} \\
Q_\theta &= -\frac{\partial M_{r\theta}}{\partial r} - \frac{1}{r}\frac{\partial M_\theta}{\partial \theta} - 2\frac{M_{r\theta}}{r} \\
&\frac{1}{r}\frac{\partial}{\partial r}(rQ_r) + \frac{1}{r}\frac{\partial Q_\theta}{\partial \theta} = -\omega^2 \rho w
\end{aligned}
\tag{3.1}
$$

which express Hooke's law and the conditions of equilibrium via the d'Alembert principle. In eqns (3.1), M_r and M_θ are the radial and circumferential bending moments, $M_{r\theta}$ the twisting moment, and Q_r and Q_θ denote the radial and the circumferential shear force, respectively, with reference to a polar coordinate system r, θ.

Comparing for a given point eqns $(3.1)_{1-3}$ with the Cartesian equations (2.15), letting the r and θ directions correspond to the x_1 and x_2 directions, respectively, we identify the following relationships for D_r, D_θ, $D_{r\theta}$, ν_r and ν_θ in eqns (3.1),

$$
\begin{aligned}
D_r\nu_\theta &= D_\theta\nu_r = \nu D_{/\!/} \\
D_r &= D_{/\!/} \\
D_\theta &= (1-\nu^2)D_{\perp} + \nu^2 D_{/\!/} \\
D_{r\theta} &= (1-\nu)D_{\perp} \\
\nu_\theta &= \nu \\
\nu_r &= \nu\frac{D_{/\!/}}{(1-\nu^2)D_{\perp} + \nu^2 D_{/\!/}}.
\end{aligned}
\tag{3.2}
$$

Taking $D_+ = h_+{}^3$ and $D_- = h_-{}^3$, which corresponds to some specific norm of the deflection function w, eqns (2.13) for $D_{/\!/}$ and $D_{\perp}$ become

$$
\begin{aligned}
D_{/\!/} &= \left[\frac{\mu}{h_+{}^3} + \frac{1-\mu}{h_-{}^3}\right]^{-1} \\
D_{\perp} &= \mu h_+{}^3 + (1-\mu)h_-{}^3.
\end{aligned}
\tag{3.3}
$$

In eqn $(3.1)_6$, ρ represents the mass of material per unit area of the plate. Within the norm of the deflection function, ρ can be expressed as

$$\rho = \mu h_+ + (1-\mu)h_-. \tag{3.4}$$

Like in[4] we consider the axisymmetric plate to perform free vibrations with a mode w having a given number n ($n \geq 0$) of nodal diameters, and we assume w in the form

$$w = v(r)\cos n\theta. \tag{3.5}$$

The corresponding angular frequency ω is assumed to be given, and it is our objective to minimize the functional

$$V = \int_{R_*}^{R} [\mu h_+ + (1-\mu)h_-] r \, dr, \tag{3.6}$$

which represents the total plate volume. Here R_* denotes the inner radius of a possible annular plate ($R_* = 0$ for a full plate).

Until further notice we both consider μ, h_+ and h_- (with $h_- < h_+$) as design variables, and assume the following constraints to be prescribed

$$\begin{aligned} 0 &\leq \mu \leq 1 \\ h_{\min} &\leq h_- \leq h_{\max} \\ h_{\min} &\leq h_+ \leq h_{\max}. \end{aligned} \tag{3.7}$$

Due to the form of eqn (3.5) we can now separate variables, i.e. define θ-independent functions m_r, m_θ, $m_{r\theta}$ and q_r by the equations

$$\begin{aligned} M_r = m_r(r)\cos n\theta, \quad M_\theta = m_\theta(r)\cos n\theta \\ M_{r\theta} = m_{r\theta}(r)\sin n\theta, \quad Q_r = q_r(r)\cos n\theta \end{aligned} \tag{3.8}$$

and then rewrite five of the six equations in (3.1) as

$$\begin{aligned} m_r &= D_r\left(v'' + \nu_\theta\left(\frac{1}{r}v' - \frac{n^2}{r^2}v\right)\right) \\ m_\theta &= D_\theta\left(\frac{1}{r}v' - \frac{n^2}{r^2}v + \nu_r v''\right) \\ m_{r\theta} &= -D_{r\theta}n\left(\frac{1}{r}v' - \frac{1}{r^2}v\right) \\ q_r &= -m_r' - \frac{n}{r}m_{r\theta} - \frac{m_r - m_\theta}{r} \\ (rq_r - nm_{r\theta})' + \frac{n^2}{r}m_\theta - 2\frac{n}{r}m_{r\theta} &= -\omega^2[\mu h_+ + (1-\mu)h_-]vr. \end{aligned} \tag{3.9}$$

The normal Cauchy form of these equations is

$$\begin{aligned} v' &= p \\ p' &= \frac{m}{D_{/\!/}} - \nu\left(\frac{1}{r}p - \frac{n^2}{r^2}v\right) \\ m' &= -Q - \frac{1-\nu}{r}m + \frac{a_1 D_\perp}{r^2}p - \frac{a_2 D_\perp}{r^3}v \\ Q' &= -\frac{1}{r}Q - \frac{n^2\nu}{r^2}m - \frac{a_2 D_\perp}{r^3}p - \left\{\omega^2[\mu h_+ + (1-\mu)h_-] - \frac{a_3 D_\perp}{r^4}\right\}v \end{aligned} \tag{3.10}$$

where eqns (3.2) have been used, and the functions $m(r)$ and $Q(r)$ (the θ-independent part of the effective radial shear force) are defined as

$$\begin{aligned} m &= m_r \\ Q &= q_r - \frac{n}{r} m_{r\theta}. \end{aligned} \tag{3.11}$$

The constants a_1, a_2 and a_3 in eqns (3.10) are defined by

$$\begin{aligned} a_1 &= (1-\nu)(1+\nu+2n^2) \\ a_2 &= n^2(1-\nu)(3+\nu) \\ a_3 &= n^2(1-\nu)(2+n^2+\nu n^2). \end{aligned} \tag{3.12}$$

The Hamiltonian of the problem is now given by

$$\begin{aligned} H = &-[\mu h_+ + (1-\mu)h_-]r + \lambda_v p + \lambda_p\left[\frac{m}{D_{\not{}}} - \nu\left(\frac{1}{r}p - \frac{n^2}{r^2}v\right)\right] \\ &+ \lambda_m\left[-Q - \frac{1-\nu}{r}m + \frac{a_1 D_\perp}{r^2}p - \frac{a_2 D_\perp}{r^3}v\right] \\ &+ \lambda_Q\left[-\frac{1}{r}Q - \frac{n^2\nu}{r^2}m - \frac{a_2 D_\perp}{r^3}p - \left\{\omega^2\left[\mu h_+ + (1-\mu)h_-\right] - \frac{a_3 D_\perp}{r^4}\right\}v\right] \end{aligned} \tag{3.13}$$

where λ_v, λ_p, λ_m and λ_Q are Lagrangian multipliers.

4. THE NECESSARY CONDITIONS OF OPTIMALTY

The problem formulated in Section 3 is self-adjoint. Then the Lagrangian multipliers of the Hamiltonian H in eqn (3.13) are proportional to the basic variables, and we have

$$\begin{aligned} \lambda_Q &= c^2 r v \\ \lambda_m &= c^2 r p \\ \lambda_p &= -c^2 r m \\ \lambda_v &= -c^2 r Q, \end{aligned} \tag{4.1}$$

where c^2 is a positive constant. Substituting the functions λ_Q, λ_m, λ_p and λ_v into eqn (3.13) and taking the variations of H with respect to h_+ and h_-, respectively, with eqns (3.3) taken into account, we obtain the following necessary conditions† for stationarity over h_+ and h_-,

$$3\mu h_+^2\left[\left(\frac{m}{h_+^3}\right)^2 + \frac{a_1}{r^2}p^2 - 2\frac{a_2}{r^3}pv + \frac{a_3}{r^4}v^2\right] - \mu\omega^2 v^2 = \frac{\mu}{c^2} \tag{4.2}$$

$$3(1-\mu)h_-^2\left[\left(\frac{m}{h_-^3}\right)^2 + \frac{a_1}{r^2}p^2 - 2\frac{a_2}{r^3}pv + \frac{a_3}{r^4}v^2\right] - (1-\mu)\omega^2 v^2 = \frac{1-\mu}{c^2}, \tag{4.3}$$

which are valid for $h_{min} < h_+ < h_{max}$ and $h_{min} < h_- < h_{max}$, respectively. From eqns (4.2) and (4.3) we are able to draw the important conclusion that the design variables h_+ and h_- cannot both be intermediate at a given point where $0 < \mu < 1$.

We now derive the necessary condition for stationarity of H (and thus V) with respect to

†These conditions may be checked if we set $\mu = 1$ or $\mu = 0$, which implies a reduction of the problem to having h_+ and h_-, respectively, as the only design variable in a traditional optimal design formulation where isotropic plate bending rigidity is assumed; for $\mu = 1$ and $\mu = 0$, respectively, eqns (4.2) and (4.3) are easily verified to coincide with the necessary stationarity condition in[4].

the design variable μ, and obtain

$$(h_+^3 - h_-^3)\left[\frac{m^2}{h_+^3 h_-^3} + \frac{a_1}{r^2} p^2 - 2\frac{a_2}{r^3} pv + \frac{a_3}{r^4} v^2\right] + (h_+ - h_-)\omega^2 v^2 = \frac{h_+ - h_-}{c^2} \tag{4.4}$$

for $0 < \mu < 1$.

To determine the type of an extremum (max/min), consider the conditions of the Weierstrass type. For the design variable h_+, the Weierstrass condition requires that

$$\begin{gathered} N \leq 0, \\ N \equiv H(h_+ + \delta h_+) - H(h_+) = \\ -\mu r c^2 \left(\frac{\delta h_+}{h_+}\right)^2 \frac{1}{(h_+ + \delta h_+)^3} \left\{ m^2 \left[6 + 8\frac{\delta h_+}{h_+} + 3\left(\frac{\delta h_+}{h_+}\right)^2\right] \right. \\ \left. + B h_+^2 (h_+ + \delta h_+)^3 (3h_+ + \delta h_+) \right\}. \end{gathered} \tag{4.5}$$

The latter expression contains only nonlinear terms in δh_+, the linear ones having vanished due to the stationarity condition. The Weierstrass condition for the design variable h_- only differs from eqn (4.5) by the factor $(1-\mu)$ instead of μ, and by h_- and δh_- instead of h_+ and δh_+, respectively.

The factor B in eqn (4.5) is given by

$$B = p\left(\frac{a_2}{r^3} v - \frac{a_1}{r^2} p\right) + v\left(\frac{a_2}{r^3} p - \frac{a_3}{r^4} v\right). \tag{4.6}$$

Consider now some special cases of Ineq. (4.5). For $n = 0$ (i.e. for an axisymmetric vibration mode), we have $a_2 = a_3 = 0$, see eqns (3.12), and $B = -a_1 p^2/r^2 \leq 0$. For those points where the radial bending moment m is small, Ineq. (4.5) breaks down, which means that the intermediate regime of h_+ is non-optimal. The same conclusion is true for the case where $\delta h_+ \gg 1$ because the term $Bh_+^2(\delta h_+)^4$ will then be prevailing within the brackets {} of (4.5). The same is also true, of course, for the design variable h_-.

The design variables h_+ and h_- clearly enter the optimization problem quite in the same way as the design variable h itself enters the initial problem of Section 1 before reqularization. According to what has been stated above and the fact that the effectivity of any small material element increases progressively with the distance by which it is disposed from the plate mid plane, we set h_+ equal to its limiting value

$$h_+ = h_{max}. \tag{4.7}$$

It follows from our discussion above that in problems where small δh_- is ensured via a comparatively small ratio h_{max}/h_{min} between the specified constraint values, h_- may be intermediate at points where the radial bending moment m is large. In problems with comparatively small h_{max}/h_{min} ratio, it is therefore recommended that both h_- and μ be used as design variables, which implies that optimality conditions (4.3) and (4.4) must both be included in the formulation for optimal design.

From now on, we direct our main interest to problems associated with a comparatively large h_{max}/h_{min} ratio. Although intermediate values of h_- can also not be excluded in particular subregions of the plate in such problems in general, we in the following set h_- equal to its limiting value,

$$h_- = h_{min}. \tag{4.8}$$

Hence, we *remain with the function $\mu(r)$ as the only design variable in the sequel.* This design variable is evidently singular (μ appears linearly in the Hamiltonian), and the necessary condition of optimality with respect to μ is therefore given by Kelley test

$$L \equiv \frac{\partial}{\partial\mu}\frac{d^2}{dr^2}\frac{\partial H}{\partial\mu} \geq 0. \tag{4.9}$$

Calculations show that

$$L \equiv c^2 r \frac{D_+ - D_-}{D_+ D_-}\left\{4mv(h_+ - h_-)\omega^2 + 2(D_+ - D_-)\left[Q\left(\frac{a_2 v}{r^3} - \frac{a_1 p}{r^2}\right) + (D_+ + D_-)\left(\frac{a_2 v}{r^3} - \frac{a_1 p}{r^2}\right)^2 - m\left(\frac{a_4 v}{r^4} - \frac{a_5 p}{r^3}\right) - m^2 \frac{a_1}{r^2}\frac{D_+ + D_-}{D_+ D_-}\right]\right\}, \tag{4.10}$$

where a_1, a_2 and a_3 are given by (3.12), and a_4 and a_5 are defined as

$$a_4 = n^2(1-\nu)(7 + 11\nu + 4\nu^2 + 2n^2 + 4\nu n^2)$$
$$a_5 = (1-\nu)(4\nu + 4\nu^2 + 9n^2 + 11\nu n^2). \tag{4.11}$$

Definite conclusions can be made concerning the sign of L in the neighbourhood of a clamped ($v = p = 0$) boundary or a free ($m = Q = 0$) boundary of a plate. For the first case, Ineq. (4.9) is not satisfied, which implies that *circumferential stiffeners (with* $0 < \mu < 1$*) will never appear at a clamped edge; the plate will always be solid with* $h = h_+$ ($= h_{max}$) *or* $h = h_-$ ($= h_{min}$) *in the vicinity of such an edge.* For the second case, Ineq. (4.9) is satisfied. However, this does not, in general, ensure that stiffeners will arise near a free edge. This is because we may have $\mu = 0$, i.e. $h = h_-$ ($= h_{min}$), and (4.9) is not valid in subintervals of constrained μ.

5. SOLUTION PROCEDURE

Since our problem is nonlinear and quite complex, closed form solutions cannot be expected, and we therefore proceed to solve it numerically by successive iterations. As it is slightly simpler to construct a solution procedure for the dual problem of maximizing the angular frequency ω for given total plate volume V, we for convenience consider this formulation of the optimization problem† in the following, ans in Section 6 present our results in this context.

To outline the iterative solution procedure, we first develop and discuss a few key formulas, and then later describe the flow of the calculations. The square of the frequency ω to be maximized, can be expressed as

$$\omega^2 = \frac{\displaystyle\int_{R_*}^{R}\left\{D_{\neq}\left(v'' + \frac{\nu}{r}v' - \frac{\nu n^2}{r^2}v\right)^2 + D_\perp(1-\nu)\left[2n^2\left(\frac{1}{r}v\right)'^2 + (1+\nu)\left(\frac{1}{r}v' - \frac{n^2}{r^2}v\right)^2\right]\right\}r\,dr}{\displaystyle\int_{R_*}^{R}\left[\mu h_{max} + (1-\mu)h_{min}\right]v^2 r\,dr} \tag{5.1}$$

if we multiply eqn $(3.9)_5$ by v, use eqns $(3.9)_{1-4}$, (3.2), $(3.11)_1$ and (4.7), and perform two integrations by parts over the interval $R_* \leq r \leq R$, assuming linear, homogeneous boundary conditions to be given. In eqn (5.1), the numerator identifies half the potential energy, and ω^2 multiplied by the denominator is half the kinetic energy of the vibrating plate. Equation (5.1) thus provides a convenient basis for applying the finite element method to solve the plate vibration problem described by eqn (3.9), i.e. to determine v and its derivatives subject to given plate design. The distance between R_* and R is subdivided by a large number of equally spaced

†A solution to the geometrically constrained problem of minimizing V for fixed ω, can always be obtained by solving a sequence of maximum ω fixed V problems, where the designs are successively scaled to meet prescribed ω.

nodal points where the deflection v and the slope $p = v'$ are used as the nodal unknowns, and where continuity is imposed on these quantities. Each element is taken to have an individual, constant value of μ, and the shape function for the deflection v is chosen as a complete third order polynomium. The stiffness matrices corresponding to $D_{\parallel}$ and $D_{\perp}$, respectively, and the mass matrix corresponding to $[\mu h_{max} + (1-\mu)h_{min}]$ are easily established on the basis of relevant expressions contained in eqn (5.1).

The optimality condition (4.4), which is valid for $0 < \mu < 1$, can be used to determine the value of μ for each element in an average sense. In view of eqns (4.7), (4.8), (3.11), $(3.9)_1$, $(3.2)_2$, $(3.2)_5$ and $(3.3)_1$, we can write eqn (4.4) as

$$\frac{(h^3_{max} - h^3_{min})h^3_{max}h^3_{min}}{(\mu h^3_{min} + (1-\mu)h^3_{max})^2}\left(v'' + \frac{\nu}{r}p - \frac{\nu n^2}{r^2}v\right)^2 + (h^3_{max} - h^3_{min})\left(\frac{a_1}{r^2}p^2 - 2\frac{a_2}{r^3}pv + \frac{a_3}{r^4}v^2\right) - (h_{max} - h_{min})\omega^2 v^2 = \frac{h_{max} - h_{min}}{c^2}. \tag{5.2}$$

Denoting the radii of the nodes of a particular element by r_{i-1} and r_i, respectively, multiplying (5.2) by rc^2 and integrating over the interval $r_{i-1} \le r \le r_i$, we obtain

$$c^2\left\{\frac{E_1}{(\mu h^3_{min} + (1-\mu)h^3_{max})^2} + E_2 - E_3\right\} = E_4, \tag{5.3}$$

where

$$\begin{aligned} E_1 &= (h^3_{max} - h^3_{min})h^3_{max}h^3_{min}\int_{r_{i-1}}^{r_i}\left(v'' + \frac{\nu}{r}p - \frac{\nu n^2}{r^2}v\right)^2 r\,dr \\ E_2 &= (h^3_{max} - h^3_{min})\int_{r_{i-1}}^{r_i}\left(\frac{a_1}{r^2}p^2 - 2\frac{a_2}{r^3}pv + \frac{a_3}{r^4}v^2\right)r\,dr \\ E_3 &= \omega^2(h_{max} - h_{min})\int_{r_{i-1}}^{r_i} v^2 r\,dr \\ E_4 &= (h_{max} - h_{min})(r_i^2 - r_{i-1}^2)/2. \end{aligned} \tag{5.4}$$

We now divide through eqn (5.3) by E_4, and, in order to be able to obtain convergence, we take the sth root of the equation and multiply through by μ. Hence, we have

$$\mu = c^{2/s}g, \tag{5.5}$$

where g is defined by

$$g = f^{1/s} \tag{5.6}$$

for $f \ge 0$, and f is given by

$$f = \frac{\mu^s}{\mu h^3_{min} + (1-\mu)h^3_{max}}\frac{E_1}{E_4} + \mu^s\frac{E_2 - E_3}{E_4}. \tag{5.7}$$

By means of eqn (5.5) that governs unconstrained values of μ, and the constraints (2.1), we can now write the following formula for μ,

$$\mu = \begin{cases} 1 & \text{if } c^{2/s}g \ge 1; \quad r \in r_{ca} \\ c^{2/s}g & \text{if } 0 < c^{2/s}g < 1; \quad r \in r_u \\ 0 & \text{if } f \le 0; \quad r \in r_{cb}. \end{cases} \tag{5.8}$$

However, in order to obtain convergence, it turned out to be necessary to introduce move-limits in the iterative procedure. Hence, instead of eqn (5.8), the following formula was actually used for determining μ,

$$\mu = \begin{cases} \text{Min}\{(1+k)\mu^*, 1\} & \text{if} \quad c^{2/s}g \geq \text{Min}\{(1+k)\mu^*, 1\}; \quad r \in r'_{ca} \\ c^{2/s}g & \text{if} \quad \text{Max}\{(1-k)\mu^*, 0\} < c^{2/s}g < \text{Min}\{(1+k)\mu^*, 1\}; \quad r \in r'_u \\ \text{Max}\{(1-k)\mu^*, 0\} & \text{if} \quad c^{2/s}g \leq \text{Max}\{(1-k)\mu^*, 0\}; \quad r \in r'_{cb} \end{cases} \tag{5.9}$$

Here, the parameter $k > 0$, and the symbols μ^* on the right-hand side denote values of μ obtained by the previous iteration. Substituting (5.9) into the volume constraint (3.6), we derive the following expression for the constant $c^{2/s}$ in eqn (5.9),

$$c^{2/s} = \frac{V - \int_{r'_{ca} \cup r'_{cb}} [\mu^* h_{\max} + (1-\mu^*)h_{\min}] r \, dr - h_{\min} \int_{r'_u} r \, dr}{(h_{\max} - h_{\min}) \int_{r'_u} g(r) r \, dr} \tag{5.10}$$

The formula (5.9) is seen to coincide with (5.8) for large values of the parameter k. For $k = 0$, the formula would remain reproducing the initial μ function. In the examples to be presented later, values of k from 0.7 to 1.5 together with values of s taken from 0.8 to 1.0 in problems with $n = 0$, and from 0.6 to 0.7 in problems with $n = 4$, were found to ensure stable (and not too slow) convergence towards the solutions.

The basic iteration scheme applied in the numerical solution procedure can now be outlined. It has the following form:

START Take $\mu(r)$ arbitrarily, and let the subinterval(s) r'_u be non-empty.
I Compute $D_{\not}$ and $D_{\perp}$ by eqns (3.3).
II Solve the plate vibration problem for given $D_{\not}$ and $D_{\perp}$, i.e. determine ω^2 and the functions v, $v' = p$ and v''. Compute $E_1, \ldots, E_4$ for each plate element by eqns (5.4).
III Determine $c^{2/s}$ by eqn (5.10), and the function $g(r)$ by means of eqns (5.7) and (5.6).
IV Determine $\mu(r)$ together with subintervals r'_{ca}, r'_u and r'_{cb} by eqn (5.9).
V Go to III if $\mu(r)$ has not converged in the inner iteration loop III–V.
VI Go to I if $\mu(r)$ and hence all other iterates have not converged in the main iteration loop I–VI.
END

6. NUMERICAL RESULTS

We now present and discuss some numerical solutions to the new optimal design formulation developed in Sections 2–4 for axisymmetric plates. Constant maximum and minimum allowable values, $h_{\max}$ and $h_{\min}$, respectively, are assumed for the plate thickness, and the concentration $\mu(r)$ of thin, circumferential stiffeners is used as the only design variable. The problems are considered in the form of maximizing the angular frequency ω corresponding to a transverse vibration mode with n ($n \geq 0$) nodal diameters for prescribed total plate volume V, and the solution procedure of Section 5 is applied.

In the following examples, the ratio between the plate thickness constraints is taken to be comparatively large, namely $h_{\max}/h_{\min} = 5$. Poisson's ratio ν of the plate material is chosen to be $\nu = 0.25$. The frequency ω of each optimal plate will in the sequel be given in proportion to the corresponding frequency ω_u of a uniform, solid reference plate that has the same material volume, plate radius (or radii) and boundary conditions. The thickness of the uniform reference plate is denoted by h_u, and is clearly a measure for the volume prescribed for the optimal plate.

Figure 2 shows results associated with $n = 4$ for annular, axisymmetric plates, whose inner radius is one-fifth of the outer radius. For these plates, $h_u/h_{\min} = 1.6579$. The edges of the optimal plate in Fig. 2(a) are both clampted, while the plate in Fig. 2(b) has clamped inner edge and simply supported outer edge. In Figs. 2(a and b), the distance between the dashed lines represents the minimum plate thickness $h_{\min}$, and the sum of the vertical distances from the dotted lines to the solid curves illustrates the concentration $\mu(r)$ of integral stiffeners (of total

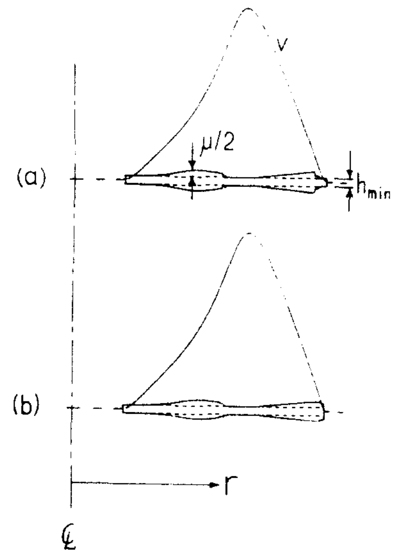

Fig. 2. Axisymmetric, annular plates optimized for $n = 4$. (a) plate with clamped inner and outer edge, $\omega/\omega_u = 1.668$; (b) plate with clamped inner edge and simply supported outer edge, $\omega/\omega_u = 1.911$. For both plates, $R_*/R = 0.2$, $h_{max}/h_{min} = 5$, $h_u/h_{min} = 1.6579$ and $\nu = 0.25$.

height $h_{max} - h_{min}$, see Fig. 1 with $h_+ = h_{max}$ and $h_- = h_{min}$). It is found that $\mu < 1$ throughout, that is, the plates have no solid subregions with thickness h_{max}. Solid subregions with thickness h_{min} ($\mu = 0$) are easily identified in the figures. The solid curve above each plate in Fig. 2 shows the θ-independent part $v(r)$ of the deflection function.

The frequency of the doubly clamped, optimal plate in Fig. 2(a) is found to be given by $\omega/\omega_u = 1.668$, i.e. it is increased by 66.8% relative to the doubly clamped, uniform reference plate. The optimal $n = 4$ frequency of the clamped-simply supported plate in Fig. 2(b) is given by $\omega/\omega_u = 1.911$, and is thus 91.1% higher than the corresponding frequency of the clamped-simply supported, uniform reference plate. These results (as well as those to be presented later) are obtained by using 50 elements in the numerical solution procedure, but it is worth emphasizing that only absolutely negligible changes are found if any other reasonable number of elements are used.

Solving precisely the same constrained problems by means of the traditional formulation for optimal design, we found lack of such invariance of the frequencies and corresponding designs on the number of elements used in the solution procedure, and the large values stated above for the optimal frequencies could also not be achieved on the basis of the traditional formulation, even if we applied a considerable number of elements. These facts clearly indicate that our new formulation provides a regularization of optimization problems associated with large h_{max}/h_{min} ratios.

In Section 4, we were able to deduce that the plates will always be solid with $h = h_{min}$ ($\mu = 0$) or $h = h_{max}$ ($\mu = 1$) in the vicinity of a clamped edge. The designs in Fig. 2 are seen to be in perfect agreement with this theoretical result, which is further illustrated in Figs. 3(a–e). Here, annular plates of the same volume, material, h_{max}, h_{min} and outer radius R are optimized with respect to the fundamental natural frequency ($n = 0$, axisymmetric mode). The plates in Fig. 3 all have free outer edge and clamped inner edge, and their only dissimilarity is their different inner radii, R_*. The radii R_* of the clamped inner plate edges in Figs. 3(a–e) are given by $R_*/R = 0.2$, 0.3, 0.4, 0.5 and 0.6, respectively. We see that while the designs 3a and b have $\mu = 0$ and hence $h = h_{min}$, the designs 3c, d and e have $\mu = 1$, and thus solid subregions of thickness $h = h_{max}$, in the vicinities of the clamped edges. An intermediate value of μ at the clamped edge could also not be obtained for any other value of R_* considered in the present example, which clearly illustrates and confirms the theoretical result derived in Section 4.

Figure 4 shows full circular plates ($R_* = 0$) optimized with respect to the fundamental frequency ω, assuming the corresponding mode to be axisymmetric, $n = 0$. The edges $r = R$ of the plates in Fig. 4 are (a) simply supported, (b) clamped and (c) free, the latter plate being

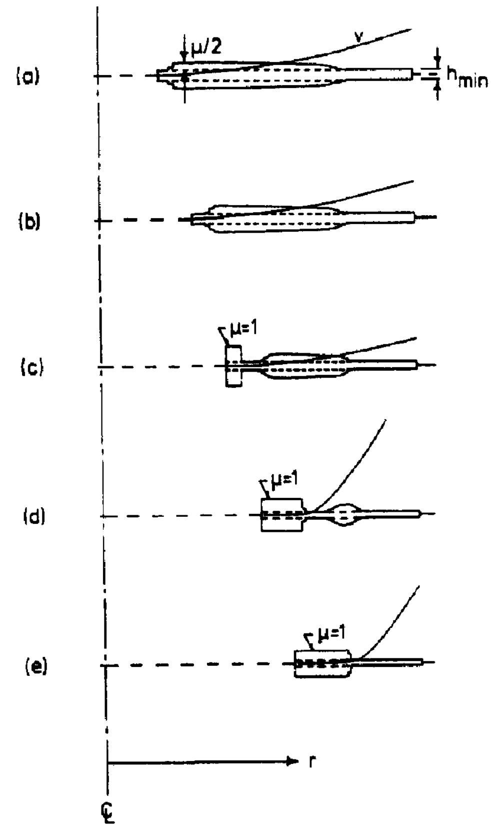

Fig. 3. Axisymmetric, annular plates optimized for $n = 0$. Inner plate edges are clamped, outer edges free. The plates all have the same volume, material ($\nu = 0.25$), h_{max}, h_{min} ($h_{max}/h_{min} = 5$) and outer radius R, but different inner radii R_0. (a) $R_0/R = 0.2$, $\omega/\omega_u = 1.85$; (b) $R_0/R = 0.3$, $\omega/\omega_u = 1.55$; (c) $R_0/R = 0.4$, $\omega/\omega_u = 1.30$; (d) $R_0/R = 0.5$, $\omega/\omega_u = 1.16$; (e) $R_0/R = 0.6$, $\omega/\omega_u = 1.24$. (The thickness scale used in the figure is decreased uniformly from (a) through (e).)

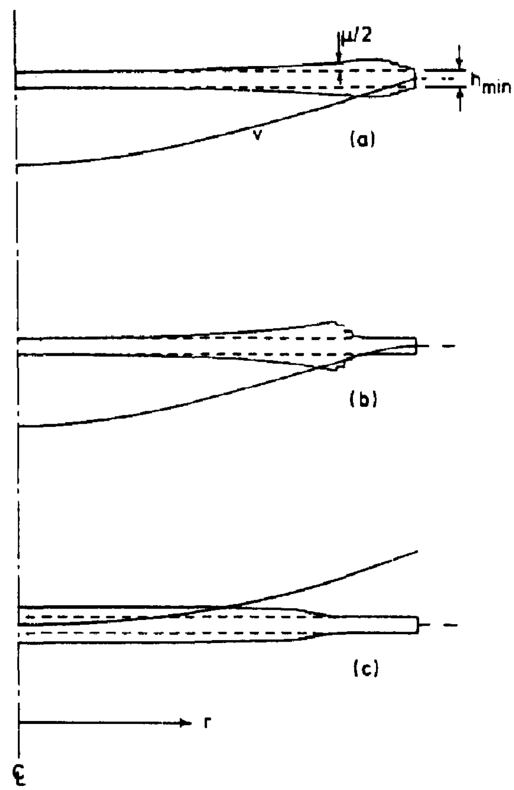

Fig. 4. Full axisymmetric plates optimized for $n = 0$. (a) simply supported plate, $\omega/\omega_u = 1.726$; (b) clamped plate, $\omega/\omega_u = 1.139$; (c) centrally supported plate with a free edge, $\omega/\omega_u = 2.390$. For the plates shown, $h_{max}/h_{min} = 5$, $h_u/h_{min} = 1.5915$ and $\nu = 0.25$.

supported at its center. Again, $h_{max}/h_{min} = 5$, $\nu = 0.25$, and we have taken $h_u/h_{min} = 1.5915$. The concentration μ of integral stiffeners is found to be smaller than unity everywhere.

The fundamental frequency of the simply supported, optimal plate in Fig. 4(a) is found to be given by $\omega/\omega_u = 1.726$. This result witness the superiority of the new optimal design formulation proposed in this paper by being a good deal higher than the result $\omega/\omega_u = 1.16$ for the geometrically unconstrained, smooth design earlier obtained in[1] and reproduced in[2] for a simply supported circular plate on the basis of the traditional optimal design formulation.

For the clamped solution in Fig. 4(b), we find the optimal frequency ω to be given by $\omega/\omega_u = 1.139$, and the optimal frequency of the centrally supported plate with a free edge in Fig. 4(c) is given by $\omega/\omega_u = 2.390$. The frequencies of these geometrically constrained designs are both lower than those associated with smooth, geometrically unconstrained designs† obtained in[1] on the basis of the traditional formulation for optimal design, namely $\omega/\omega_u = 1.53$ for a clamped plate and $\omega/\omega_u = 6.44$ for a centrally supported plate with a free edge. However, the latter frequencies can easily be exceeded if we solve the new formulation for larger h_{max}/h_{min} ratios, and hence make a more relevant comparison.

7. DISCUSSION

During the last years it has become clear that many problems concerning optimal design or optimal distribution of material properties of continuous media, possessed no solutions within some initial formulation[7, 8, 10, 13–15]; in the problems considered in[14, 15], it is required to distribute the material characteristics of some nonhomogeneous medium within the body such as to extremize some functional of the solution to the corresponding boundary value problem.

In[13], a problem concerning optimal distribution of specific resistances within a MHD-power generator channel is investigated as an example, and some physical reasons are given for the absence of an optimal isotropic control for this problem. It is essential that any solution to the problem, including the optimal one, is characterized at each point not only by the amplitudes of the corresponding vectors (current density and potential gradient), but also by the directions of these vectors. The optimal medium should in essence reflect its dependence upon these directions, which means that it should be anisotropic. If the initial formulation of the problem only allows isotropic media for comparison, the optimal control turns out to be characterized by an infinity of zones occupied by isotropic materials of different properties, which is nothing but a complete analog of a sliding regime[16]. For the physical problems in question, the sliding regime of control is just the same as some special kind of anisotropic medium.

Formally speaking, the absence of the optimal solution to such problems is intimately connected with the impossibility to satisfy the Weierstrass necessary condition at almost each point of the region[13]. This condition presents just an instrument which enables us to outline the necessity of regularization. That is, a necessity of some special closure of the set of admissible controls.

Consider the set of solutions $z(u)$ to the boundary value problem consisting of the differential equation

$$L(u)z = 0$$

and given boundary conditions, where $u \in U$ represents the control function of the set U.

Let $\{z_i\} = \{z(u_i)\}$ be some arbitrary weakly convergent sequence of solutions. Denote by z^0 the weak limit of such a sequence:

$$z_i \rightharpoonup z^0.$$

The function z^0 need not correspond to some control $u \in U$. Determine, following[17], G-closure of the set U of functions $\bar{u}$ such that for any z^0 there exists an element $\bar{u}$ which can

†These designs are obtained for $\nu = 0.3$, but the influence of ν is not significant.

be set into correspondence

$$z^0 = z^0(\bar{u})$$

with the element z^0. Then $U \cup \bar{U}$, where $\bar{U} = \{\bar{u}\}$, turns out to be G-closed, and the problem of minimization of some weakly continuous functional $I(z)$ of the solution z possesses its solution within this set. The minimum value $I(z^0)$ of the functional is then the limit of some minimizing sequence $I(z_i)$,

$$I(z^0) = \lim_{i\to\infty} I(z^i).$$

For the problems considered in the present paper, the absence of a global optimal solution within the traditional formulation is due to the fact that the set of istropic controls, which is usually considered as being admissible, turns out not to be G-closed in general. When we construct the minimizing sequence of isotropic controls, we arrive at rapidly varying sequences of inclusions of different properties, which tend towards some anisotropic medium. The properties of such media depend both upon the properties of the compounds and on the geometry of the inclusions themselves.

For the second order problem

$$\text{Min}\, I(z);$$
$$\int_\Sigma u \operatorname{grad} z \operatorname{grad} \eta \, dx = \int_\Sigma \eta f(x)\, dx, \quad \forall \eta \in W_2^1(\Sigma)$$
$$f(x) \in L_1(\Sigma), \quad u \in L_\infty(\Sigma), \quad u_- \le \text{vrai max}\, u \le u_+$$

where the functional $I(z)$ is semicontinuous from below and Σ represents some domain in R^n, the G-closure of the set of controls was found in[18] and[19]. The $\bar{U}$ set has turned out to be that of symmetric matrices (symmetric tensors of the second rank), whose eigenvalues $\lambda_1, \ldots, \lambda_n$ disposed in an ascending order are connected by the inequalities

$$u_- \le \lambda_1 = \frac{u_+ u_-}{u_+ + u_- - \lambda_2} \le \lambda_2 = \lambda_3 = \ldots = \lambda_n \le u_+.$$

The tensors of such kind describe the equivalent properties of a composite consisting of a rapidly varying sequence of layers of initial isotropic material components.

For the problem considered in this paper, the G-closure of the set of admissible controls is also provided by a layered structure. It is a mere consequence of the assumed axisymmetry of the control, i.e. its dependence on the radial coordinate only. Therefore, the layered compounds represent here the G-closure of the set of controls.

Passing over to some more than one-dimensional problems of optimization of properties of elastic bodies, we find that the problem of G-closure is fairly more complicated. The difficulty is connected with the necessity of describing the microstructure and equivalent (smear-out) characteristics of cellular structures which arise due to infinitely many inclusions of one material into another. Problems of this type have been considered in several papers, of which mention should be made of[20–24]. In these papers are given the recipes which would make it possible to calculate the equivalent tensor of elastic moduli corresponding to some given periodic microstructure of the composite material. For the regularized optimization problem, the components of that equivalent tensor are to be treated as controls, and these components will actually depend on the shape and relative sizes of the initial materials within a cell.

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